AIAA Journal

VOLUME 7 | OCTOBER 1969 | NUMBER 10

An Improved Differential Approximation for Radiative Transfer with Spherical Symmetry

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An improved closure condition is used to construct a purely differential equation that, with associated boundary conditions, describes radiative transfer with spherical symmetry. The method is equivalent in degree of complexity to the P_4 moment approximation, that is, a spherical harmonic expansion retaining four terms. The improved closing condition is especially constructed to allow both isotropic radiation and unidirectional, beamlike radiation. The method is applied to the problem of radiative equilibrium of a grey gas between two concentric black spheres maintained at different temperatures. Both radiative heat flux and temperature distribution from the present model are compared to known exact results. Predictions from the improved differential approximation in the crucial limit of a transparent gas are superior to those from both the conventional P_2 and P_4 moment approximations, which predict qualitatively incorrect results in this limit when the inner sphere becomes small compared to outer.

I. Introduction

THIS study deals with curvature effects in radiative transfer for a grey gas under conditions of local thermodynamic equilibrium. Even with these simplifications transfer theory is so complicated that exact results are available only in one-dimensional geometries, either plane-parallel or with spherical or cylindrical symmetry. There do exist approximate methods. One widely used approximation is the spherical harmonics method. As formulated, this theory has no explicit restrictions on either opacity or geometry, hence it would seem to be a promising candidate for geometrically complicated radiation fields such as those of a re-entry vehicle or a cylinder of hot gas in a shock tube of finite radius. It turns out, however, that as a practical matter the method fails whenever the directional distribution of radiation becomes singular, or when any volume element in the gas is not traversed by photons coming from all directions. Such would be the case for a radiation field far from a source, or in the space between concentric spheres or cylinders. To my knowledge, no manageable method exists which will allow radiative heat transfer calculations for arbitrary geometry and opacity.

In this paper an approximation is given which adequately describes radiative transfer with spherical symmetry. The contribution of the walls to the radiation field is not restricted to be small. The method is a modification of the spherical harmonics method, and therefore a brief attempt at a perspective on this method is in order.

The spherical harmonics method can be carried out to various degrees of approximation. In radiative transfer, the first approximation has become known as the differential approximation;[1] in the closely related field of neutron transport theory it is known as the diffusion approximation[2,3] and it is a generally satisfactory method for plane-parallel problems. Despite earlier warnings that the diffusion approximation fails for problems with curvature when the mean free path becomes comparable to the radius of curvature (Ref. 2, p. 219), one still finds the hope expressed that the differential approximation may be used without geometrical restrictions (Viskanta,[4] Cheng and Vincenti[5]). It was shown by Cess[6] and later by Dennar and Sibulkin[7] that the method fails for the case of radiative equilibrium between concentric cylinders or spheres. In the transparent limit, with small inner cylinder or sphere compared to outer, the inner radiation heat flux is overpredicted by a factor of two. This error is much larger than what is found with this approximation in plane-parallel problems, which is typically of the order of 10%. Cess blames the difficulty on the part of the radiation field contributed by the walls; however, recently Chisnell[8] has obtained the same overestimate for the case of a gas confined between cold concentric spheres. It appears that there are serious curvature difficulties in the differential approximation which do not involve wall emission.

Higher approximations would seem to be the logical road to improvement. The next approximation is called the P_4 approximation, because four terms are kept in the spherical harmonic expansion of the directional distribution of intensity. Two are kept in the first approximation. Unfortunately, no such calculation is reported in the radiation literature for problems with curvature. The plane-parallel

Received December 12, 1968; revision received March 3, 1969. Work supported by the U. S. Air Force Office of Scientific Research, under Contract AF49(638)-1726. This investigation was completed while on leave in the Department of Thermal Engineering, Cornell University. The author is grateful to F. K. Moore for this visit and his continued insistence on a unidirectional beam of radiation as a necessary feature of any reasonable approximation.

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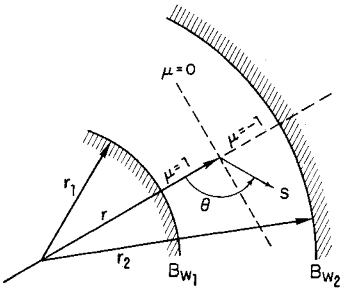

Fig. 1 Geometry and nomenclature.

case does indeed improve, as shown by Le Sage[9] and Sherman.[10] There the next approximation, using either full range or the equivalent half-range moments, gives about an order of magnitude increase in accuracy. With curvature, some trouble can be saved by a glance at the neutron transport literature. In an early study Davison[11] reports that the convergence of the spherical harmonic method is so slow for a sphere small compared to mean free path that satisfactory results have not yet been attained with the P_8 approximation. Thus higher spherical harmonics approximations appear not to help.

Perhaps for this reason other schemes have been proposed for spherical or cylindrical radiation problems. One is due to Chou and Tien,[12] as improved by Hunt.[13] This method is an adaptation of the Lees moment method in kinetic theory, in which the directional distribution of intensity is taken to be several different averaged values in several solid angle regions. This method gives reasonable results, but it has the disadvantage that the "shadow" cast by the inner sphere or cylinder with respect to the various lines of sight from some position in the annular gap is specifically built into the method. Thus generalization to another geometry appears very difficult. Another method has been proposed by Olfe.[14,15] Based on the reasoning that it is radiation from the walls which is responsible for failures of the differential approximation in the transparent limit, Olfe splits the radiation into two parts and treats only that from gas emission by the differential method. Wall radiation is taken from the exact solution of the corresponding problem. This also works quite well in the concentric sphere problem, but of course it requires the exact expression for the wall contribution. This is by no means trivial in a spherical geometry and is not available in general. Olfe uses the exact results of Ryhming[16] for concentric spheres. It is not clear how the cold wall failure discussed by Chisnell would be handled by this technique. In such a case, Lee and Olfe offer yet another method[17] that involves numerical iteration on certain exact integral equations for spheres, starting with the differential approximation.

The methods just discussed share the common feature that the rational spherical harmonics expansion procedure is abandoned in favor of special assumptions appropriate to a particular situation, or recourse to special exact solutions. One can argue, however, that there is nothing wrong in principle with spherical harmonics for problems with curvature, and that the difficulty is one of insufficiently rapid convergence. The method is related to another expansion method, that of Case.[18] Case's method is a powerful but complicated eigenfunction expansion for neutron transport which can lead to both exact and more rapidly converging approximate results. It has been applied to a plane-parallel radiating gas slab by Ferziger and Simmons[19] with improved results compared to the differential approximation. It is claimed by Nonnemacher[20] that Case's expansion is identical to the P_n approximation in the limit $n \to \infty$. The idea thus arises that spherical harmonics could still become useful if one were to supply a little help.

In this paper the approximate closure condition specified by the P_4 harmonics expansion with spherical symmetry is modified to allow for a singular directional distribution of intensity. The modification permits relatively simple analytical solutions that predict the proper trends for the case of radiative equilibrium between concentric spheres. The same form is retained for the required closure condition as that given by the spherical harmonics method. It is hoped that this use of certain results of a spherical harmonics expansion as a guide may prove useful for a corresponding modification valid for arbitrary geometries.

II. P_2 and P_4 Approximations

The spherically symmetric geometry and nomenclature is defined with the help of Fig. 1. Two concentric spheres, outer of radius r_2 and inner of radius r_1, contain a grey gas with volume absorption coefficient K. The walls emit isotropically; their temperature T is described by B_{w_2} and B_{w_1}, respectively, with $B = \sigma T^4/\pi$. With symmetry around the radial direction the intensity of radiation I varies only with r and $\mu = \cos\theta$, where θ is measured from the inward direction. Then with the assumption of local thermodynamic equilibrium the radiative transfer equation is (see Ryhming[16] and note the change in definition of θ)

$$\mu \frac{\partial I}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu} = K(I - B) \tag{1}$$

Directional moments are defined in the same way as for a plane-parallel geometry, since there is only one preferred direction, by

$$I_n = 2\pi \int_{-1}^{1} I\mu^n d\mu$$

Multiplying Eq. (1) by $\mu^n (n \geq 0)$ and integrating over all directions gives a set of moment equations,

$$\frac{dI_{n+1}}{dr} + \frac{(n+2)}{r} I_{n+1} - \frac{n}{r} I_{n-1} = K\left\{I_n - \left[\frac{(1)^n + (-1)^n}{n+1}\right] 2\pi B\right\} \tag{2}$$

These equations are still exact but indeterminate if they are written out for any finite n. I_0 is the average intensity and I_1 is the radial radiative heat flux, positive in the inward direction.

The spherical harmonics approximation works as follows. In the Nth approximation I is expanded in Legendre polynomials with coefficients A_m, keeping a finite number of terms,

$$I = \sum_{m=0}^{2N-1} A_m P_m(\mu)$$

The coefficient A_{2N} is set equal to zero. Using Eqs. (2), one gets $2N$ ordinary differential equations for $2N$ of the A's. But these coefficients are related to the moments, and thus one finds a corresponding determinate set of moment equations. An equivalent procedure is to work only with the moments and use as a closing relation that relation between moments which results from $A_{2N} = 0$. The result for the P_2 approximation is

$$\begin{aligned} dI_1/dr + 2I_1/r &= K(I_0 - 4\pi B) \\ dI_2/dr + 3I_2/r - I_0/r &= KI_1, \; I_2 = I_{0/3} \end{aligned} \tag{3}$$

The P_4 approximation consists of the first two of Eqs. (3) and in addition of

$$\begin{aligned} dI_3/dr + 4I_3/r - 2I_1/r &= K(I_2 - 4\pi B/3) \\ dI_4/dr + 5I_4/r - 3I_2/r &= KI_3 \\ 35I_4 - 30I_2 + 3I_0 &= 0 \end{aligned} \tag{4}$$

These differential equations are exact. The last equation in both Eqs. (3) and (4) is an algebraic closing relation that makes the remaining differential equations determinate. Since this is the only place where an approximation has been made, this closing condition becomes the key to the theory. With it the P_2 approximation consists of two ordinary differential equations for two unknowns; the P_4 approximation has four. These can be combined into a single equation for one unknown. With absorption coefficient K variable this becomes awkward, and from now on K will be taken as constant. This assumption has no effect on the validity of the method since the closing conditions does not contain K. Defining $\tau = Kr$ as a new independent variable, and choosing I_0 as a dependent variable, one gets for the first approximation

$$\nabla^2 I_0 - 3(I_0 - 4\pi B) = 0 \tag{5}$$

The second approximation is

$$\nabla^4 I_0 - 10\nabla^2 I_0 + \tfrac{35}{3}(I_0 - 4\pi B) + \tfrac{55}{9}\nabla^2(4\pi B) = 0 \tag{6}$$

Equations (5) and (6) can also be obtained from a spherical harmonics expansion without any geometrical restrictions. Giovanelli[21] gave the first approximation for radiative transport, the second approximation was given by Traugott.[22] For neutron transport in arbitrary geometries the corresponding equations are given by Davison,[3] including the next higher approximation.

The variable of greatest interest is usually not the average intensity I_0 but the radiative heat flux I_1. This is obtained from I_0 with the help of the first of Eqs. (3). Substituting this into Eq. (5) or (6) and integrating the result once gives, for the first approximation

$$I_1 = (d/d\tau)(I_0/3) \tag{7}$$

For the second

$$I_1 = \frac{d}{d\tau}\left[\frac{4\pi B}{3} + \frac{6}{7}(I_0 - 4\pi B) - \frac{3}{35}\nabla^2 I_0\right] \tag{8}$$

A constant of integration that would otherwise appear in Eq. (8) has been dropped for consistency with the case of radiative equilibrium, $I_0 = 4\pi B$. In that case Eq. (7) gives the well-known Rosseland limit, and Eq. (8) the next approximation to it. In an opaque, plane-parallel geometry, using a Taylor series expansion on B, one gets the correct limit,

$$I_1 = \frac{d}{d\tau}\left[\frac{4\pi B}{3} + \frac{d^2}{d\tau^2}\left(\frac{4\pi B}{5}\right)\right] \tag{9}$$

This equation has been given by Shen.[23]

A fundamental deficiency of the spherical harmonics expansion becomes apparent for a spherical geometry. In Eqs. (5) and (6) the spherical forms of the Laplacian can be written as

$$\nabla^2 \ldots = (1/\tau)(d^2/d\tau^2)(\tau \ldots)$$
$$\nabla^4 \ldots = (1/\tau)(d^4/d\tau^4)(\tau \ldots)$$

Thus the P_2 and P_4 approximations can be transformed to the corresponding plane-parallel problem. The same holds true in the next approximation, as can be seen from the form of the corresponding equation in Davison,[3] and presumably it holds for all approximations. For a sphere there does in fact exist a corresponding transformation of the exact integral expressions (Heaslet and Warming,[24] Ambarzumian[25]) but for a spherical shell there does not (Heaslet, private communication). In a certain sense the spherical harmonics method ignores the inner sphere. In the next section a remedy is given which retains the basic framework of the P_4 approximation.

III. Modified Closing Condition

At the heart of the spherical harmonics method are algebraic closing relations of the kind given by the last of Eqs. (3) or (4). These relations owe their coefficients to the orthogonality properties of Legendre polynomials. Since the coefficients are constants, the relations hold for any one-dimensional geometry whether plane, cylindrical, or spherical. An interesting physical interpretation is possible when one notices that these relations are identically true for isotropic radiation. One could obtain the P_2 approximation in a more intuitive way by invoking the isotropic relation between second and zeroth moments; this is then just the Milne-Eddington approximation. Since only one constant appears, only one physical argument is allowed. The P_4 closing condition, however, has two constants. This allows another condition to be incorporated. It is now assumed that an algebraic relation between I_4, I_2 and I_0 is a reasonable approximation, and that the relation should be true for isotropic radiation and, in addition, for the most nonisotropic radiation possible, namely, a unidirectional beam. This involves the singular directional distribution for which the spherical harmonics method fails. With spherical symmetry the beam is radial and one finds, with I a δ function at $\mu = \pm 1$, $I_0 = I_2 = I_4$. For isotropy $I_0 = 3I_2 = 5I_4$. Both conditions are satisfied with

$$5I_4 - 6I_2 + I_0 = 0 \tag{10}$$

With this relation instead of the last of Eqs. (4), one finds that the first four moment equations reduce to

$$\frac{d^4}{d\tau^4}(\tau^2 I_0) - 6\frac{d^2}{d\tau^2}(\tau^2 I_0) + 5\tau^2(I_0 - 4\pi B) + \frac{13}{3}\frac{d^2}{d\tau^2}(\tau^2 4\pi B) + \frac{10}{3}\frac{d}{d\tau}(\tau 4\pi B) = 0 \tag{11}$$

Compared to Eq. (6) one gets not only different coefficients for the various terms but the form of the equation has changed. It cannot be reduced by a transformation to the plane-parallel case. The heat flux is given, from the same procedure as before, by

$$I_1 = \frac{d}{d\tau}\left(\frac{4\pi B}{3}\right) + \frac{6}{5\tau^2}\frac{d}{d\tau}[\tau^2(I_0 - 4\pi B)] - \frac{1}{5\tau^2}\frac{d^3}{d\tau^3}(\tau^2 I_0) \tag{12}$$

This also reduces to Eq. (9) for an opaque plane-parallel geometry, but with curvature the extra terms are different from those derivable from Eq. (8).

Before proceeding with the solution of these equations it is necessary to discuss boundary conditions. This is done in the next section.

IV. Boundary Conditions

It has often been suggested that one has a choice of either Mark's or Marshak's boundary conditions with the spherical harmonics method (Davison[3]). As pointed out below, however, consistency requires use of Mark's boundary condition. For the first approximation one can easily show that Marshak's boundary condition is consistent with a two-stream expansion using half-range moments. Mark's boundary conditions must be modified for consistency with the modified differential equation. First, the conventional results are given.

The walls are taken to emit isotropically; hence $I = B_w$ over a hemisphere based on the surface. Mark's condition is approximate; it consists of satisfying the above relation in discrete directions given by $P_{2N}(\mu) = 0$ in the Nth approximation.

For the P_2 approximation one finds, for $\tau = a$ and b, respectively,

$$[(3)^{1/2}I_1 \pm (I_0 - 4\pi B_w)]_{a,b} = 0 \tag{13}$$

Here the upper sign refers to the outer sphere, $\tau = a$.

For the P_4 approximation the Mark boundary conditions are found to be two algebraic equations connecting the first four moments. These can be put into the form

$$\begin{aligned}[3.194I_2 \pm 3.836I_1 + 0.936I_0]_{a,b} &= 8\pi B_{wa,b} \\ [\pm 8.074I_3 + 7.730I_2 - 0.577I_0]_{a,b} &= 8\pi B_{wa,b}\end{aligned} \tag{14}$$

Since no geometrical considerations have been used, the constant coefficients in Eqs. (14) are the same in any one-dimensional geometry. These coefficients are again fixed by constants which appear in various Legendre polynomials. The coefficients will now be determined instead from the limiting conditions of isotropy and a radial beam, preserving the mathematical form of the equations. These conditions are not enough, however. In contrast to the closing condition, Eq. (10), equations such as Eqs. (14) are not homogeneous. Three constants will have to be determined in each equation, from three conditions. Because by choice the coefficients are independent of geometry, one can use as an extra condition the requirement of consistency with the following general plane-parallel integral of Eq. (11):

$$I_0 - 4\pi B = -\frac{\pi}{3}\left\{\int_{-\infty}^{\tau} [5e^{-(5)^{1/2}(\tau-t)} + e^{-(\tau-t)}]\frac{dB}{dt}\,dt - \int_{\tau}^{\infty} [5e^{-(5)^{1/2}(t-\tau)} + e^{-(t-\tau)}]\frac{dB}{dt}\,dt\right\} \tag{15}$$

This is the general solution for an arbitrary temperature distribution with no wall emission. For the case of a slab, bounded on the left of $\tau = b$ and to the right of $\tau = a$ by either a vacuum or a cold black wall, dB/dt will be a δ function at the boundaries and the expression above becomes

$$I_0 = 4\pi B - \frac{\pi}{3}\left\{B_b[5e^{-(5)^{1/2}(\tau-b)} + e^{-(\tau-b)}] + B_a[5e^{-(5)^{1/2}(a-\tau)} + e^{-(a-\tau)}]\right\} - \frac{\pi}{3}\left\{\int_b^{\tau} [5e^{-(5)^{1/2}(\tau-t)} + e^{-(\tau-t)}]\frac{dB}{dt}\,dt - \int_{\tau}^{a} [5e^{-(5)^{1/2}(t-\tau)} + e^{-(t-\tau)}]\frac{dB}{dt}\,dt\right\}$$

From the moment equations one then obtains, by integration, corresponding expressions for the other moments. From these expressions, evaluated at the boundaries, one finds that the moments there are related to each other as follows, again with the upper sign for $\tau = a$:

$$\begin{aligned}\{3[5 - (5)^{1/2}]I_2 \pm 12I_1 + 3[(5)^{1/2} - 1]I_0\}_{a,b} &= 0 \\ \{\pm 12(5)^{1/2}I_3 + 3[5(5)^{1/2} - 1]I_2 - 3[(5)^{1/2} - 1]I_0\}_{a,b} &= 0\end{aligned} \tag{16}$$

These then are boundary conditions which have been derived from the modified differential equation in this special case without wall emission. They have the same form as Eqs. (14), and one also notices that they are true for a beam either outwards or inwards, $I_0 = \mp I_1 = I_2 = \mp I_3$. They will become true as well for isotropic radiation in equilibrium with an emitting wall by adding $2[(5)^{1/2} + 1]4\pi B_{wa,b}$ to the right of Eqs. (16). This leads to the present boundary conditions, which are two equations of the same form as Eqs. (14) with the modified three coefficients of the first given by

$$\frac{3[5 - (5)^{1/2}]}{(5)^{1/2} + 1} = 2.562, \quad \frac{12}{(5)^{1/2} + 1} = 3.708$$

$$\frac{3[(5)^{1/2} - 1]}{(5)^{1/2} + 1} = 1.146$$

The coefficients of the second equation are

$$\frac{12(5)^{1/2}}{(5)^{1/2} + 1} = 8.292, \quad \frac{3[5(5)^{1/2} - 1]}{(5)^{1/2} + 1} = 9.438, \quad \frac{3[(5)^{1/2} - 1]}{(5)^{1/2} + 1}$$

The same procedure just described, used with either the conventional P_2 or P_4 approximation, leads to Mark's boundary conditions. The method above is therefore an alternate way of deriving these same boundary conditions from a solution such as Eq. (15) of the differential equation, rather than from consideration of discrete directions related to roots of Legendre polynomials. It is in this sense that Mark's rather than Marshak's boundary conditions are consistent with full-range spherical harmonics.

The use of compatible boundary conditions is essential in obtaining proper limiting results. The present boundary conditions are only as accurate as the modified differential approximation, even though better ones can easily be obtained. A slightly more accurate set is given below for completeness; it will be compatible with half-range moment expansion methods. Use with Eq. (11) for either spherical or plane-parallel slabs leads to nonsense.

Both isotropy and a beam satisfy $3I_2 - 4I' + I_0 = 0$, $8I^3 - 9I_2 + I_0 = 0$. Here I^1 and I^3 are differences in half-range moments, namely,

$$I^n = 2\pi\int_0^1 I\mu^n d\mu - 2\pi\int_{-1}^0 I\mu^n d\mu$$

At either surface one has exactly

$$I_a{}^n = 4\pi B_{wa}/(n + 1) - I_{na}$$

$$-I_b{}^n = (-1)^n 4\pi B_{wb}/(n + 1) - I_{nb}$$

This then gives as more accurate boundary conditions

$$\begin{aligned}[3I_2 \pm 4I_1 + I_0]_{a,b} &= 8\pi B_{wa,b} \\ [\pm 8I_3 + 9I_2 - I_0]_{a,b} &= 8\pi B_{wa,b}\end{aligned} \tag{17}$$

Equations (17) are incompatible with Eq. (11).

It is convenient to express the boundary conditions in terms of I_0 only, since only this variable appears in the governing differential equation. Thus for the P_2 approximation one obtains from Eq. (13) a well-known result, often interpreted by means of a "linear extrapolation distance":

$$[(dI_0/d\tau) \pm (3)^{1/2}(I_0 - 4\pi B_w)]_{a,b} = 0 \tag{18}$$

For the modified differential approximation one gets after much manipulation with the moment equations

$$\begin{aligned}&\left[f_1\frac{d^2I_0}{d\tau^2} + f_2\frac{dI_0}{d\tau} + f_3(I_0 - 4\pi B_w) - \frac{13}{3}f_1(I_0 - 4\pi B)\right]_{a,b} = 0 \\ &\left[f_1\frac{d^3I_0}{d\tau^3} + f_4\frac{dI_0}{d\tau} + \frac{13}{3}f_1\frac{d(4\pi B)}{d\tau} - f_5(I_0 - 4\pi B_w) + \frac{14f_1}{\tau}(I_0 - 4\pi B)\right]_{a,b} = 0\end{aligned} \tag{19}$$

where

$$f_1 = \left[1 \mp \frac{12}{5[(5)^{1/2} - 1]\tau}\right]_{a,b}$$

$$f_2 = \frac{4}{5 - (5)^{1/2}}\left[\pm(5)^{1/2} + \frac{4 - (5)^{1/2}}{\tau} \mp \frac{9}{(5)^{1/2}\tau^2}\right]_{a,b}$$

$$f_3 = \frac{4}{5 - (5)^{1/2}}\left[\frac{(5)^{1/2} + 25}{6} \mp \frac{1}{\tau} + \frac{2}{[5 - (5)^{1/2}]\tau^2}\right]$$

$$f_4 = \frac{1}{5 - (5)^{1/2}}\left[[(5)^{1/2} - 25] \mp \frac{48}{(5)^{1/2}\tau} + \frac{6[3(5)^{1/2} - 11]}{\tau^2} \pm \frac{144}{(5)^{1/2}\tau^3}\right]_{a,b}$$

$$f_5 = \frac{4}{5-(5)^{1/2}}\left[\pm 5 + \frac{(5)^{1/2}+45}{2\tau} \mp \frac{6}{\tau^2} + \frac{12}{[5-(5)^{1/2}]\tau^3}\right]_{a,b}$$

No simple physical interpretation of these complicated boundary conditions seems possible. The terms with reciprocal powers of τ are curvature terms that disappear in the plane-parallel limit. Curvature thus explicitly appears in the boundary conditions. Note that no such terms appear in Eq. (18). In the next section, Eqs. (11, 12, and 19) are applied to a spherical shell in radiative equilibrium.

V. Radiative Equilibrium between Concentric Spheres

In the case of radiative equilibrium $I_0 = 4\pi B$ and Eq. (11) becomes an equation for the temperature distribution. It can be put into the form

$$\frac{d^4(\eta B)}{d\eta^4} + \frac{4}{\eta}\frac{d^3(\eta B)}{d\eta^3} - \frac{d^2(\eta B)}{d\eta^2} = 0$$

with $\eta = (5/3)^{1/2}\tau$. The solution is, with E_1 the first exponential integral,

$$B = C_4 + \frac{C_3}{\eta} + \frac{C_1}{2}\left[E_1(\eta) - \frac{e^{-\eta}}{\eta}\left(1-\frac{1}{\eta}\right)\right] - \frac{C_2}{2}\left[E_1(-\eta) + \frac{e^{\eta}}{\eta}\left(1+\frac{1}{\eta}\right)\right] \quad (20)$$

The four constants of integration are determined by Eqs. (19), rewritten in η, and using $(I_0)_{a,b} = 4\pi B_{a,b}$. Then in each equation the last term disappears and the next to the last term becomes a temperature slip term. Once the constants are known the heat flux is fixed because one obtains from Eq. (12)

$$I_1 = \frac{4\pi}{3}\left(\frac{5}{3}\right)^{1/2}\left[\frac{dB}{d\eta} - \frac{1}{\eta^2}\frac{d^3(\eta^2 B)}{d\eta^3}\right] = -\frac{4\pi}{3}\left(\frac{5}{3}\right)^{1/2}\frac{C_3}{\eta^2} = (I_1)_b\frac{b^2}{\tau^2} \quad (21)$$

Because of the complicated structure of Eqs. (19), it is tedious to obtain the constants for any opacity although there is no difficulty in principle. For this reason only transparent and opaque limiting results are given. The transparent limit is the crucial test because it is there that the conventional spherical harmonics method breaks down. Equation (20), of course, is valid for any opacity.

Transparent Limit

When τ and therefore η is small, one expands Eq. (20) to get

$$B = C_4 + D_3/\eta + D_1[\eta + \eta^3/60 + 0(\eta^5)] + D_2[1/\eta^2 + (\tfrac{3}{2} - \gamma) - \ln\eta - \eta^2/24 + 0(\eta^4)] \quad (22)$$

where $\gamma = 0.5772$ and these new integration constants are related to the old ones by

$$C_1 = 3D_1 + D_2,\ C_2 = 3D_1 - D_2,\ C_3 = 6D_1 + D_3$$

The constants are obtained by substituting into Eq. (19), being careful to consistently keep terms up to order η^4 compared to unity in the final equations. This is necessary because both of Eqs. (19) turn out to be identical to the order η^3. The logarithmic term from the expansion of $E_1(\eta)$ cancels out. One finds that $C_4 = 0(B_w)$, $D_1 = D_2 = D_3 = 0(a^2B_w)$. Thus only the first and last term in Eq. (22) contribute to the temperature distribution, whereas D_1 and D_3 determine the heat flux. The final result, for $b < \tau < a \ll 1$, is

$$B = \frac{a^4B_{wa} + b^4B_{wb}}{a^4+b^4} - \frac{(B_{wa}-B_{wb})}{3[(5)^{1/2}-1]}\frac{a^2b^2}{\tau^2}\frac{(a^2-b^2)}{a^4+b^4} \quad (23)$$

$$(I_1)_b = \frac{4}{3[(5)^{1/2}-1]}\pi(B_{wa} - B_{wb})\frac{a^2(a^2+b^2)}{a^4+b^4} \quad (24)$$

Corresponding exact results are, from Ryhming,[16]

$$B = B_{wa}\left[\frac{\tau + (\tau^2-b^2)^{1/2}}{2\tau}\right] + B_{wb}\left[\frac{\tau - (\tau^2-b^2)^{1/2}}{2\tau}\right] \quad (25)$$

$$(I_1)_b = \pi(B_{wa} - B_{wb}) \quad (26)$$

The P_2 or differential approximation gives

$$B = (a^2B_{wa} + b^2B_{wb})/(a^2+b^2) \quad (27)$$

$$(I_1)_b = \frac{2}{(3)^{1/2}}\pi(B_{wa} - B_{wb})\frac{2a^2}{a^2+b^2} \quad (28)$$

The conventional P_4 approximation in the transparent limit results in the same emissive power as given by Eq. (27), whereas the heat flux has the same form as that given by Eq. (28) with the factor $2/(3)^{1/2} = 1.1547$ replaced by 1.0422.

Neither of these results is correct, as discussed in the next section. This lack of improvement with the P_4 method verifies the slow convergence of the spherical harmonics method for radiation when the directional distribution becomes singular, just as with neutron transport.

Opaque Limit

In this case, let $\eta \to \infty$ in Eq. (20). This leaves

$$B = C_4 + C_3/\eta + C_1e^{-\eta}/2\eta^2 - C_2e^{\eta}/2\eta^2$$

Again, these constants come from Eqs. (19), which become in this limit just a condition that there be no temperature slip at the walls. The final results are that the last two terms become smaller than the first two, and

$$B = \frac{a}{\tau}B_{wa}\left(\frac{\tau-b}{a-b}\right) + \frac{b}{\tau}B_{wa}\left(\frac{a-\tau}{a-b}\right) \quad (29)$$

$$(I_1)_b = \frac{4\pi}{3}\frac{a}{b}\frac{(B_{wa}-B_{wb})}{(a-b)} \quad (30)$$

Both results are exact in this limit, again see Ryhming.[16] It is known that the P_2 approximation also gives the correct opaque limit (see, for instance, Olfe[15]), and a P_4 calculation turns out to have the same property, as might have been expected.

VI. Discussion of Results

Before discussing the effects of curvature, it is useful to consider first the plane-parallel limit. Though not a crucial test for the differential methods, this situation is nevertheless not trivial. One lets $b \to \infty$ keeping $\eta_1 = (5/3)^{1/2}(\tau - b)$ fixed, and then all curvature terms disappear. Without curvature terms in the boundary conditions, Eqs. (19), it is relatively easy to obtain results without assuming either small or large opacity. One integrates $d^4B/d\eta_1^4 - d^2B/d\eta_1^2 = 0$. This gives $B = A_1e^{\eta_1} + A_2e^{-\eta_1} + A_3\eta_1 + A_4$. The heat flux now is constant, with $I_1 = (4\pi/3)(5/3)^{1/2}A_3$.

The form of the solution is the same with the P_4 approximation, the only difference being the appearance of a different independent variable $\xi_1 = [(35)^{1/2}/3](\tau - b)$ instead of η_1. The P_2 approximation is linear for all opacities, as is well known. The exact solution is linear only in the transparent and opaque limits, thus either the P_4 approximation or the

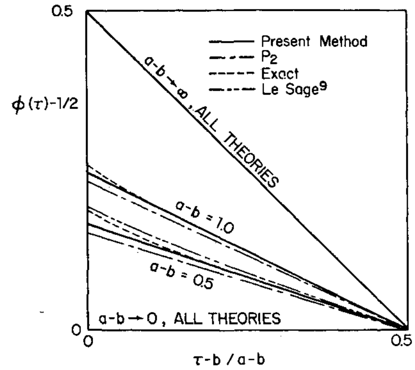

Fig. 2 Emissive power of a plane-parallel slab.

present method approximates this departure from linearity with additional exponential terms. A numerical comparison of the various methods for two intermediate opacities, $a - b = 1.0$ and 0.5, is given in Fig. 2. The dimensionless variable $\varphi(\tau) = (B_{w_a} - B)/(B_{w_a} B_{w_b})$ —has been used. The exact emissive power distribution is available from the numerical study of Heaslet and Warming.[26] The other results are from the present method, the P_2 approximation, and the two-term half-range calculation of Le Sage[9] which is roughly equivalent to the P_4 approximation. Figure 2 shows, as is already known, that even the P_2 approximation does not do badly in the absence of wall curvature effects. Higher approximations are desirable but not crucial, and it does not matter precisely which scheme is used. The contribution of the exponential terms here is not large.

In the thin and thick planar limits one finds the same results as are obtained from Eqs. (23, 24, 27, and 28) by letting $b \to a$. The resulting expressions tend to the proper exact limits with the exception of the transparent heat fluxes which are listed in Fig. 4.

Comparison to exact Eq. (26) shows the presence of spurious factors slightly larger than unity. In a plane-parallel geometry this error can easily be traced. The general plane-parallel heat flux corresponding to Eq. (15) is

$$I_1 = 2\pi \left\{ \int_{-\infty}^{\tau} \left[\frac{(5)^{1/2}}{9} e^{-(5)^{1/2}(\tau - t)} + \frac{1}{6} e^{-(\tau - t)} \right] \frac{dB}{dt} dt + \int_{\tau}^{\infty} \left[\frac{(5)^{1/2}}{6} e^{-(5)^{1/2}(t-\tau)} + \frac{1}{6} e^{-(t-\tau)} \right] \frac{dB}{dt} dt \right\}$$

When this is compared to its exact counterpart, one finds the latter to have the same form but with integrands replaced by E_3, the third exponential integral. Comparing these functions for vanishing argument gives a value of $[(5)^{1/2} + 1]/6$ to be compared to $E_3(0) = \frac{1}{2}$, and this ratio is $[(5)^{1/2} + 1]/3 = 4/3[(5)^{1/2} - 1] = 1.0787$. Similar explanations can be found for the other constants, which are thus error contributions from an exponential approximation to an exponential integral. These spurious factors can be eliminated by using half-range methods. However, standard half-range methods are useful only for problems without curvature. An attempt to modify Le Sage's half-range method in the same spirit as the present P_4 modification was made but proved unsuccessful.

With curvature, only the transparent limit is of interest because of the critical blockage effect of the inner sphere, as mentioned earlier. The opaque limit is exact and any finite opacity tends to make the various differential approximations better. Both the P_2 approximation, Eq. (27), as well as the P_4 approximation give a constant temperature in the transparent limit when, in fact, a function of τ is required. The present method, Eq. (23), does better. This is shown in Fig. 3 for four values of b/a. For $b/a \to 1.0$ all methods are correct. For $b/a \to 0$ one gets, using again the dimensionless variable $\varphi(\tau)$, for the present method

$$\varphi(\tau) = \frac{1}{3[(5)^{1/2} - 1)]} \left(\frac{b}{\tau}\right)^2$$

The corresponding exact result is

$$\varphi(\tau) = \tfrac{1}{2}\{1 - [1 - (b/\tau)^2]^{1/2}\}$$

Except in the immediate neighborhood of the inner sphere this is approximately $\varphi(\tau) = \frac{1}{4}(b/\tau)^2$.

The accuracy of the approximation is thus quite good away from the inner sphere, since again the factor $4/3[(5)^{1/2} - 1]$ occurs, and this is nearly unity. At the inner sphere itself the relative slip in emissive power is 0.2697 rather than the required 0.50.

The exact transparent heat flux to the inner sphere, from Eq. (26), is independent of the radii of the two spheres. The approximate methods contain two error contributions. One is the fixed contribution which remains in the plane-parallel limit and which has just been discussed. The other is a function of geometry. This function is plotted in Fig. 4. Both the P_2 and P_4 approximations overestimate the heat flux for an inner sphere of vanishing size, with respect to the plane-parallel case, by 100%. The present method gives a maximum error of 21% at $(b/a)^2 = (2)^{1/2} - 1$ and seems to have done very well indeed in accounting for the blocking effect of the inner sphere.

One can pinpoint fairly easily the root of the failure of the P_2 and P_4 approximations in the transparent limit to predict a temperature variation, in contrast to the present method. This is done by looking at the structure of the solutions with radiative equilibrium, $I_0 = 4\pi B$.

For the P_2 approximation Eq. (5) integrates readily to

$$B = \alpha_1 + \alpha_2/\tau \qquad (31)$$

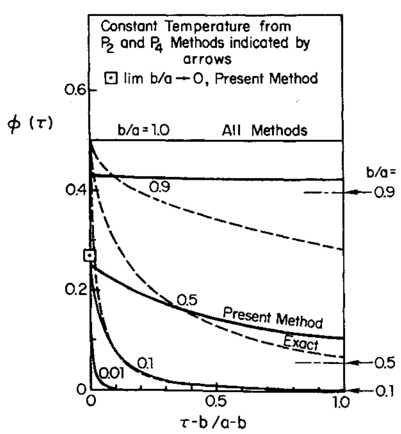

Fig. 3 Transparent emissive power of a spherical shell.

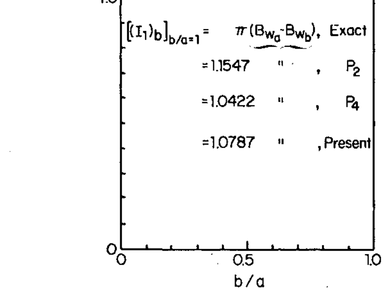

Fig. 4 Heat flux to inner sphere, transparent limit.

The heat flux is given by Eq. (7) as

$$I_1 = -(4\pi/3)(\alpha_2/\tau^2) \quad (32)$$

From Eq. (32) one infers $\alpha_2 = 0(a^2B_w)$, and then only the first term in Eq. (31) matters in the transparent limit. Thus the temperature is constant.

For the P_4 approximation integration of Eq. (6) with $\xi = [(35)^{1/2}/3]\tau$ leads to

$$B = m_4 + \frac{m_3}{\xi} + \frac{m_1}{2}\frac{e^{-\xi}}{\xi} - \frac{m_2}{2}\frac{e^{\xi}}{\xi} \quad (33)$$

The heat flux is, from Eq. (8),

$$I_1 = -(4\pi/3)[(35)^{1/2}/3](m_3/\xi^2) \quad (34)$$

Expanding Eq. (33) for small ξ gives the transparent P_4 approximation:

$$B = m_4 + (n_3/\xi) + n_1[\xi + (\xi^3/12) + 0(\xi^5)] + n_2[1 + (\xi^2/6) + 0(\xi^4)] \quad (35)$$

where $m_1 = 2n_1 - n_2$, $m_2 = -2n_1 - n_2$, $m_3 = -2n_1 + n_3$.

One can see intuitively that since from Eq. (34) m_3 must be $0(a^2B_w)$ so will n_1 and n_3; hence as in Eq. (22) only the first and least terms contribute to the temperature. But m_4 and n_2 cannot be larger than $0(B_w)$, and the temperature is again constant. Comparing Eqs. (22, 31, and 35) one traces the difficulty with the P_2 and P_4 approximation directly to the absence of a term proportional to the inverse square of the optical radius. Alternatively, the functional form of the last term in Eq. (22) permits the boundary-layer-like variation of temperature near the inner sphere shown in Fig. 2, whereas the last term in Eq. (35) does not. The first differential approximation, Eq. (31), does not even have such a term. The occurrence of large gradients near the inner sphere in the transparent limit is quite general; this can be seen from the calculations of Ryhming[16] and the more extensive numerical examples given by Viskanta and Crosbie.[27]

VII. Summary and Conclusions

A closure relation for the exact moment equations has been postulated which simultaneously satisfies both the limiting conditions of isotropic and unidirectional radiation. Its mathematical form is identical to that which comes from the P_4 spherical harmonics method. The closure condition is related to the P_4 spherical harmonics approximation in a roughly analogous fashion to the way in which the Milne-Eddington or Schuster-Schwarzschild approximations are related to the P_2 approximation, except that the coefficients of the closure relation do not seem to be derivable from any sort of systematic expansion. An improved differential approximation results with which it has been possible to reproduce known essential features of radiative equilibrium between concentric spheres. Here the usual differential approximations fail. This demonstrates the ability to handle blockage or shadow difficulties, which are associated with a singular directional distribution of intensity, with ordinary methods of analysis requiring only the integration of a differential equation and associated boundary conditions. The differential equation does not have shadow zones explicitly built into it nor does it depend on an exact solution, in contrast to other modified differential methods.[12,15]

While giving less accuracy than what has come to be expected from differential approximations in a plane-parallel geometry, the method has the advantage of simplicity and generality. For radiative equilibrium and constant absorption coefficient analytical results were obtained. With these it was possible to explore the root of the difficulties with the conventional P_2 and P_4 approximations. Neither radiative equilibrium nor the absorption coefficient is involved in the postulated closure condition; hence the method is more generally applicable to radiating gas flows with variable properties. For these more complicated problems, greater simplicity will mean much more manageable numerical computing schemes.

The present form of the closing relation requires symmetry of the intensity about a preferred direction. The same relation will therefore be applicable to any one-dimensional problem, whether plane-parallel, spherical, or cylindrical. Without this restriction, in a general multidimensional situation, it may again be possible to borrow the mathematical form of the more numerous, known closing relations of the P_4 approximation and modify these to be correct in the limit of a unidirectional beam. This is an obvious area for future work, as is application to other transport problems, both neutron and molecular.

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OCTOBER 1969 AIAA JOURNAL VOL. 7, NO. 10

Formulation of Two-Dimensional Radiant Heat Flux for Absorbing-Emitting Plane Layer with Nonisothermal Bounding Walls

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A two-dimensional approximate formulation for the radiant heat flux and its divergence is developed for the case of an absorbing-emitting medium bounded by nonisothermal parallel plates. For the optically thin limit, the present analysis approaches the rigorous exact solution. The formulation is believed to be simple enough to be useful for a variety of practical problems of radiative equilibrium and combined conduction-convection and radiation pertaining to the geometry and boundary conditions of this model. The applicability of this formulation and the "two-dimensionality" effects are demonstrated for the case of radiative equilibrium in the presence of a gray gas of constant properties.

Nomenclature

B = radiosity
C = general constant
C_p = specific heat at constant pressure
e = blackbody emissive power
E_n = exponential integral $E_n(\tau) = \int_0^1 t^{n-2} \exp\left(-\frac{\tau}{t}\right)dt$
$F(\tau,r)$ = function defined by Eq. (7)
$G(\tau,r)$ = function defined by Eq. (10)
I = intensity of radiation
q = heat-flux rate
r = the ratio X/Y
r_0 = the ratio $X/(Y_0 - Y)$
s = coordinate measured in the direction of a pencil of rays
t = dummy variable of integration
T = absolute temperature
x = optical distance in X direction $\int_0^x \kappa dX$
X = Cartesian coordinate
Y = Cartesian coordinate
θ = polar angle measured from normal
κ = absorption coefficient
μ = $\cos\theta$
σ = Stefan-Boltzmann constant
τ = optical distance in the Y direction, $\int_0^y \kappa dY$
ϕ = polar angle measured from the positive X direction
ω = solid angle

Subscripts and superscripts

0 = displacement between the walls, also used for the intensity emitted from a solid surface
1 = lower plate at $X < 0$
2 = upper plate at $X < 0$
e = upper wall at $X > 0$
r = radiation
s = in the direction of a pencil of rays
w = lower wall at $X > 0$
ν = specular
$(\ )^+$ = in the positive Y direction
$(\ )^-$ = in the negative Y direction

Received August 30, 1968; revision received March 24, 1969.
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Introduction

THE first step in solving a heat-transfer problem is to formulate the conservation of energy equation. Insofar as convection and conduction are considered, the heat flux is controlled by the local temperature and the temperature gradient, leading to a differential formulation that is solvable for proper boundary conditions.

However, in many physical situations, especially at high temperatures, one must consider also the radiant heat flux. When the medium is absorbing-emitting the formulation of the radiant heat flux enters into the energy equation, thereby introducing a serious difficulty for a solution. Contrary to conduction and convection, the radiant heat transfer is,